KEPLER'S LAWS OF PLANETARY MOTION :-

Kepler's laws of planetary motion are as follows:-

(i) The laws of elliptical orbits:- Every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.

(ii) The laws of areas :- The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal times i.e the areal velocity of the radius vector is constant.

(iii) The Harmonic law:- The square of the period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the ellipse.

Keplers laws have been enunciated purely on the basis of observations, taken for the motion of the planets. The planets move around the sun under the influence of gravitational force which is an inverse square law force. Hence, we deduce the Kepler's law of planetary motion around the sun on the basis of inverse square law of force.

Deduction of the Kepler's first law:-

for $u = \frac{1}{r}$, the inverse square law force $\left[f(r) = \frac{-K}{r^2} \right]$, is

$$f\left(\frac{1}{u}\right) = -Ku^2$$

Thus the differential equation of the ~~orbit~~ orbit can be expressed as,

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{J^2u^2} Ku^2 \quad \left[f(r) = -Ku^2 \right]$$

or, $\frac{d^2u}{d\theta^2} + u - \frac{mk}{J^2} = 0$ ——— (1)

let, $x = u - \frac{mk}{J^2}$

then, $\frac{d^2x}{d\theta^2} + x = 0$ ——— (2)

which has the solution, $x = A\cos(\theta - \theta')$ —— (3)

where A and θ' are the constants of integration.

since,

$x = u - \frac{mk}{J^2}$ and $u = \frac{1}{r}$

from eqn (2) can be written as

$$\frac{1}{r} - \frac{mk}{J^2} = A\cos(\theta - \theta')$$

or, $\frac{1}{r} = \frac{mk}{J^2} + A\cos(\theta - \theta')$ ——— (4)

or, $\frac{J^2/mk}{r} = 1 + \frac{J^2A}{mk}\cos(\theta - \theta')$

or, $\frac{l}{r} = 1 + e\cos(\theta - \theta')$ ——— (5)

Where, $\frac{J^2}{mk} = l$ and $\frac{J^2A}{mk} = e$

or, $\boxed{e = \sqrt{1 + \frac{2EJ^2}{mk^2}}}$ where e is called eccentricity.

The Magnitude of e decides the nature of the orbit.
~~If the total energy~~

values of e	values of total energy E	Conic
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	ellipse.
$e = 0$	$E = -\frac{mK^2}{2J^2}$	Circle.

Deduction of Kepler's second law :-

According to this law, the radius vector drawn from the sun to a planet, sweeps out equal areas in equal time, i.e the areal velocity is constant in planetary motion.
A gravitational force is a central field and this law is the same, as given ~~the~~ by the statement of conservation of angular momentum in a central force field.

i.e, $\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m}$, a constant. —— (6)

The inverse square law force or gravitational force in planetary motion in a special case of central force. However, the constancy of areal velocity is a general property taking place under central force.

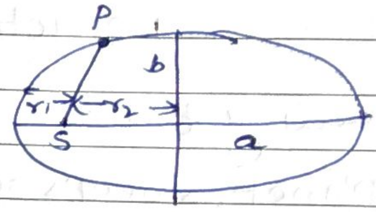
Deduction of Kepler's Third law:-

for, $e<1$ or, $E<0$, the orbit is elliptical

$$\frac{l}{r} = 1+e\cos(\theta-\theta')$$

where, $l = \frac{J^2}{mK}$ and $e = \sqrt{1+\frac{2EJ^2}{mK^2}}$

when, $\theta-\theta' = 0$, or, $\cos(\theta-\theta') = 1$



the value of $r=r_1$ is minimum

and when, $\theta-\theta' = \pi$

or, $\cos(\theta-\theta') = -1$

the value of $r=r_2$ is maximum.

The apsidal distances r_1 and r_2 are known as perihelion and aphelion.

$$\frac{l}{r_1} = 1+e \quad \text{or, } r_1 = \frac{l}{1+e} \quad \text{—— (7)}$$

$$\text{and, } \frac{l}{r_2} = 1-e \quad \text{or, } r_2 = \frac{l}{1-e} \quad \text{—— (8)}$$

The semi-major axis (a) of the ellipse is one half the sum of these two apsidal (turning) distances, i.e,

$$a = \frac{r_1+r_2}{2} = \frac{1}{2}\left[\frac{l}{1+e} + \frac{l}{1-e}\right] = \frac{l}{1-e^2} \quad \text{—— (9)}$$

or,

$$a = \frac{-J^2}{mK} \cdot \frac{mK^2}{2EJ^2} = \frac{-K}{2E} \quad \left[\text{where } 1-e^2 = \frac{-2EJ^2}{mK^2}\right] \quad \text{--- (10)}$$

or, $E = \frac{-K}{2a}$ --- (11)

Thus in case, of an elliptical orbit, the total energy depends solely on the major axis.

If 'T' be the periodic time in which the particle or radius vector completes one revolution then the area of the orbit is

$$A = \int_0^A dA = \int_0^T \left(\frac{1}{2} r^2 \dot{\theta}\right) dt = \int_0^T \frac{J}{2m} dt = \frac{JT}{2m} \quad \text{--- (11A)}$$

But area of the ellipse $A = \pi ab$. --- (12)

where a & b are the semi major and semi-minor axes of the ellipse respectively.

from equation (11A) & (12), we get–

$$T = \frac{2\pi abm}{J} \quad \text{--- (13)}$$

But, according to the property of the ellipse,

$$b = a\sqrt{1-e^2} = a\sqrt{\frac{J^2}{mKa}} = a^{1/2}\frac{J}{\sqrt{mK}} \quad \text{--- (14)}$$

from eqn (9), $a = \frac{l}{1-e^2} = \frac{J^2}{mK(1-e^2)}$

$$\text{or, } 1-e^2 = \frac{J^2}{mKa}$$

Therefore, from eqn (13) & (14), we get

$$T = \frac{2\pi a m}{J} \cdot \frac{a^{1/2} J}{\sqrt{mK}}$$

or, $$T = 2\pi a^{3/2} \sqrt{\frac{m}{K}} \quad \text{——— (15)}$$

This gives the periodic time in an elliptical orbit.

Squaring both sides of eqn (15), we get

$$T^2 = \frac{4\pi^2 a^3 m}{K}$$

or $$\boxed{T^2 \propto a^3}$$

Thus the square of period of revolution of a planet around the sun is proportional to the cube of semi-major axis of the elliptical orbit.